Alelhomology between conics

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Abstract

The goal of this communication is to find the homological transformations that are responsible for the mutual generation of any couple of conic sections.

In anthropomorphic terms, the ideas described here would correspond to the problem described as 'given two ellipses, find the position from where one ellipse is seen as the other, and viceversa'.

New procedures for geometric calculation are described, in some cases used in combination with computer algorithms.

The solution to the aforementioned problem is particularly relevant in Projective Geometry, with applications to the treatment of quadric surfaces, computer algorithms, reverse engineering, positioning systems and artificial vision.

1 Introduction

Computer algorithms have made possible the interaction between Analytics Geometry, Algebra and Projective Geometry. It has also made possible the interaction of Pluncker and Poncelet's procedures.
From Euclides until mid the 20th century, geometric loci were reduced to circumferences, parallel and perpendicular lines. It was using these that today's Geometry was built.
Having access to all kind of curves and surfaces as geometric loci for geometric research would open a wide range of possibilities, conditioning the pedagogy used to teach Geometry.

The limitations in the treatment of homology between 3D surfaces do not come from Projective Geometry, but from the widely used Descriptive Geometry approach. The systems we use to represent geometric objects are based on their apparent contours, similarly to the discrete sampling of analogical signal used in information theory. Its limitations come together with those of our visual perception.

When studying homologic transformations between quadric surfaces we need to determine the homology between two given conic surfaces: the homology that allows to determine one from another and viceversa. This mutual generative relationship is called alelhomology (ἀλλήλων, one to another, mutually). Every alelhomology consists of two transformation matrix related by an inversion. In homogeneous coordinates and in planar homology, it will be a 3x3 matrix; 4x4 in 3D.

His matrix inversion is translated in the graphical calculations by a permutation of limit-lines *k* and *'l*, or limit-planes, κ and 'λ, which have exchangeable roles in the plane or in the space.

2 Notable lines and directions

New tools need new procedures that are postulated from unknown or not previously reported (or considered) properties.

Let's describe some infinite points from the plane (directions) and some lines whose properties and tracing are worth knowing.

2.1 Tangent lines to conics through a point T

Given our impossibility to draw with precision an ellipse or a hyperbola, we have never paid attention to the following property: the radius of an ellipse of a hyperbola that passes from the center C and goes through the point T has the conjugated direction to that of the tangent t that goes through T (Cf.: fig 1)

I have never seen this property mentioned before. [1]

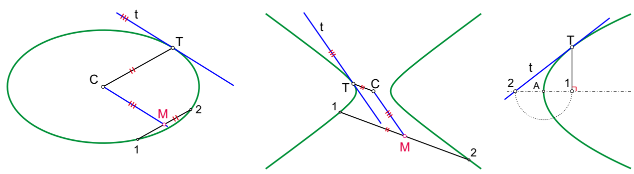


Fig. 1 Tangent through point T

So, given an ellipse and a hyperbola, to trace the tangent through T:

1) The chord 12, parallel to CT, is traced.
2) The line CM is traced through the middle point of M of the chord
3) The tangent is the line t, that goes through T parallel to CM

The case of the parabola is solved similarly knowing that the vertex is the middle point of the 12 segment.

2.2 e-diameters

D Given two conics, we called e-diameters to the diameters d and d' that are conjugated of the parallel diameters to the homology axis. These diameters are homologous and their opposite side points, if not overlapping or if they are not double-points, determine the center of homology.

From Fig. 2 two starting options arise:

- To assume that points P and Q are double points. In this case the homology axis *e-e'* is determined.
- Or to assume that P and Q overlap, but they are not double-points, which opens the possibility to calculate

another homology axis.

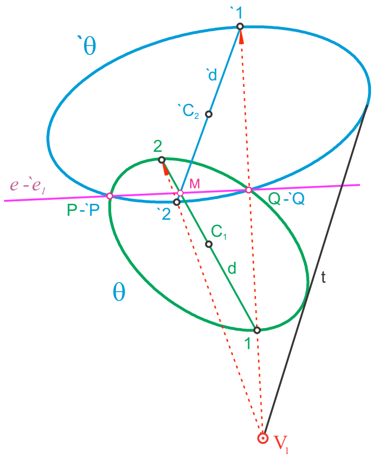


Fig. 2 Determination of V_1

1) Point M, the middle point of PQ, and C_1, determine the e-diameter d, and, analogously, M and ‘C_2 determine d.
2) Lines 1’1 and 2’2 are two V-rays that intersect in V_1, the homology center.

The tangent to θ, traced from V_1, is also tangent to`θ.

If, instead of linking with V-rays the opposite side points *external-external* and *internal-internal*, we link *external-internal* and *external-internal*, the center of homology V_2 is obtained.

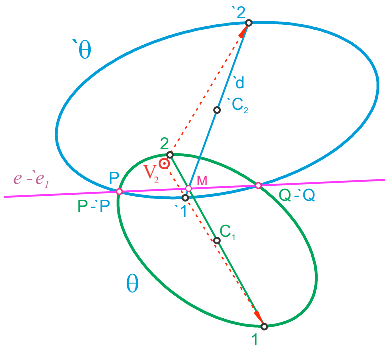


Fig. 3 V_2 *determination*

Every e-diameter contains the center of its conic and the anticenter of the homologous conic. Cfr.: Fig. 4.

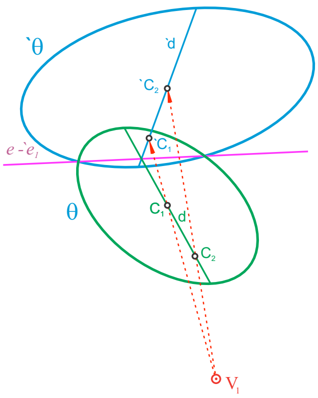


Fig. 4 Centers and anti-centers

2.3 Invariant chords

If homothetic ellipses θ and ´θ are drawn centered in C, the homothecy center, we observe that the subtended chords between two adjacent intersection points have constant directions.

The two directions are conjugated in the two families of ellipses. Their directions are invariant upon a homothetic transformation with center in C.

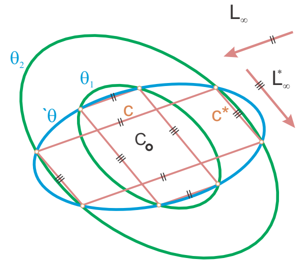


Fig. 5 Invariant infinite points

We call these chords invariant chords, because their infinite points are invariant.

These two directions, infinite points L_∞ and L^*_∞, are characteristic for each pair of given ellipses families homothetic of θ and homothetic of `θ.

The chords subtended by the opposite sides of diameters parallel to two homology axes, possible in this case, $e_1 \| e - `e_1$ and $e_2 \| e - `e_2$ are analogously subtended by the e-diameters `$d_1$ and `d_2.

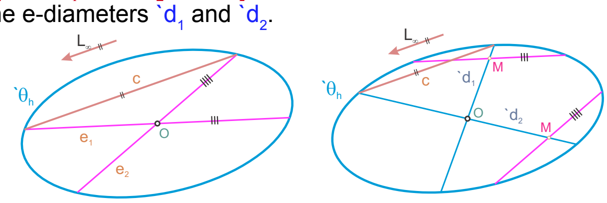


Fig. 6 Determinant chord

If L_∞ is known, given an homology axis, the other axis can be known. In this case:

$$e - `e_2 \| e_2.$$

These two invariant directions provide also the tangent points in the linkage of both families of ellipses, as shown in Fig. 7.

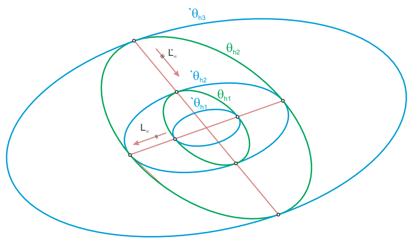


Fig. 7 Inside and outside tangecy points

2.4 Associated homothecies

V_1 and V_2 are concomitantly associated to two center of homothecy, H_1 and H_2, whose calculation provides the knowledge of the directions of both homology axes, and the invariant directions of the above-mentioned chords.

The procedure is illustrated in Fig. 8

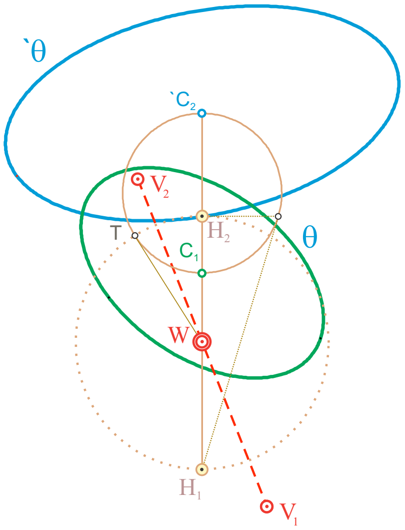


Fig. 8 H_1 *and* H_2 calculation

1)We find W, the middle point of the V_1V_2 segment, placed on the line of centers $`C_2C_1$.
2)We draw the circumference of diameter $`C_2C_1$
3)The tangent to the circumference from W provides T.
4)The circumference centered in W and with radius from W to T, intersects the line of centers in the centers of homothecy H_1 and H_2 .

In Fig. 9 we draw the ellipse θ_h, centered in $`C_2$ and homothetic of the ellipse θ, with center of homothecy H_1.

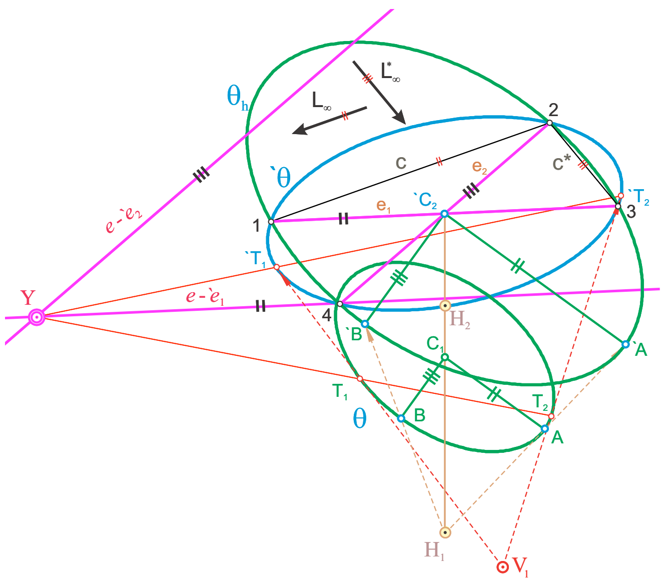


Fig. 9 Axes and chords

The homothetic ellipse θ_h intersects the image ellipse $`\theta$ determining the diameters 13 and 24, parallel to the homology axes.

The polar lines T_1T_2 and $`T_1$ $`T_2$ of V_1 with respect to θ and $`\theta$ intersect in the point Y (co-pole).

The homology axes e - $`e_1$ and $e$ - $`e_2$ are found tracing parallel to the diameters 13 and 24 from Y.

Chords c and c* determine the infinite points L_∞ and L^*_∞, that are conjugated directions with respect to the ellipse origin θ and image $`\theta$.

The same result is obtained drawing a homothetic ellipse to $`\theta$, centered in C_1.

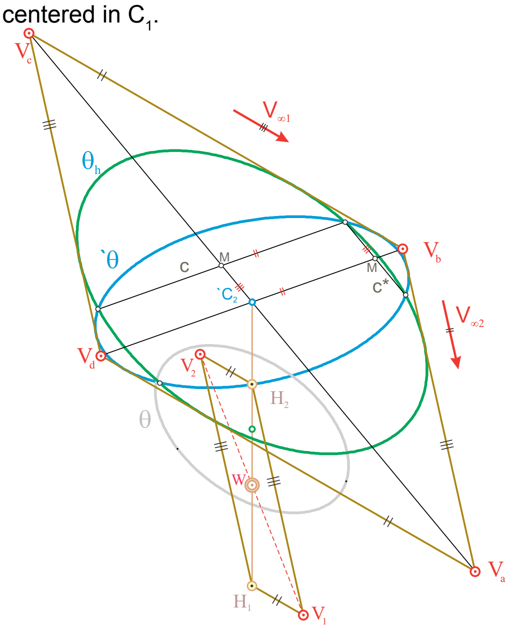


Fig. 10 Some possible homologies

In Fig. 10 some of the possible centers of homology between $`\theta$ and θ_h.

The parallelism between the sides of the parallelograms centered in W and $`C_2$. is surprising. This is due to the co-linearity of centers, required by the closed product of homologies.

2.4 Co-pole and co-polar line

The intersection point Y of the two homology axes is called co-pole, as it is the pole with respect to the two ellipses of line V_1V_2 (co-polar).

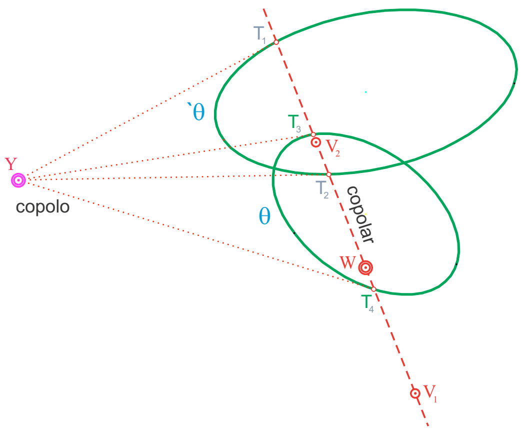


Fig. 11 Co-pole and co-polar

Polars of W intersect in Y the same way concurrent polars correspond to aligned poles.

If a V-ray intersecting two points of an ellipse is known, we count with a more efficient alternative to calculate the axes in the tangent intersection (Cfr.: Fig. 12).

We trace the tangents through points 1 and 2 and their homologous`1 and `2. The intersection points 3 and 4 of the tangents with equal parity points, determine the homology axis e - $`e_2$.

Points 5 and 6 are found analogously with the pair of tangents traced through points of different parity.

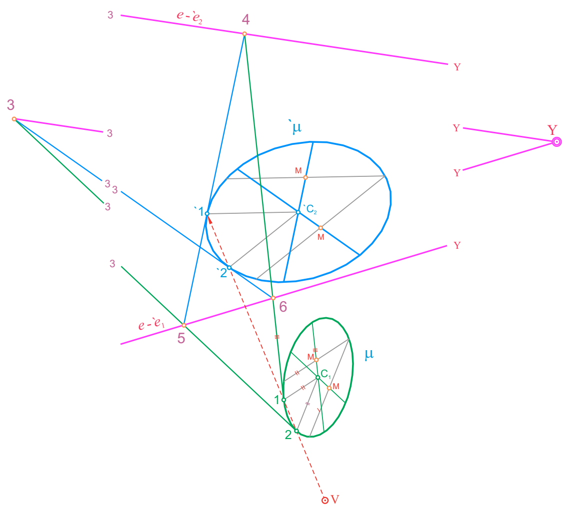


Fig. 12 Axes calculation

3. Matrix calculus

The homological transformation matrixes operating ob homogeneous coordinates, for 3D and the plane, are respectively (Cfr.: [2] pags 3.15)

$$T_{3D} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/\delta & \gamma/\delta \end{vmatrix} \qquad (1)$$

$$T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/\delta & \gamma/\delta \end{vmatrix} \qquad (2)$$

The parameter γ is the oriented distance KV_1 and the parameter δ is the oriented distance V_1F.

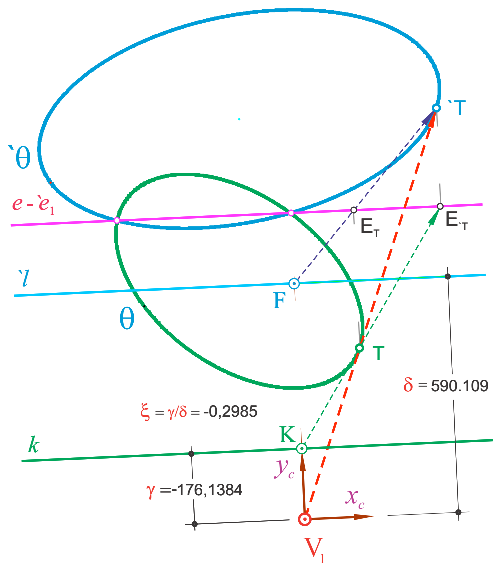


Fig. 13 Parameters γ *y* δ

The determinant of the transformation matrix is the characteristic $\xi = \gamma/\delta$ of the homology.

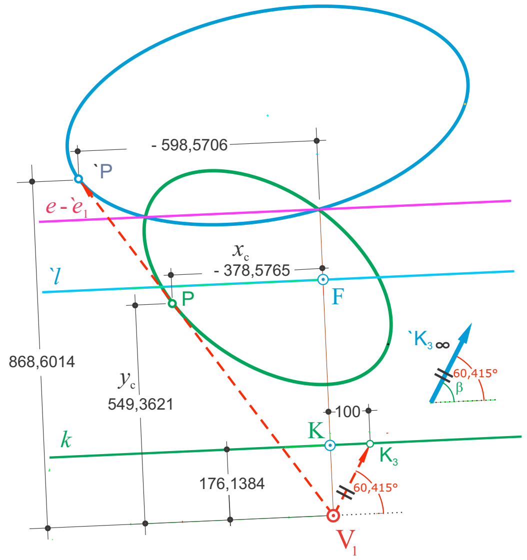


Fig. 14 Cartessian coordinates

The homogeneous coordinates of point P, known is Cartesian coordinates, are:

$$p_c = \begin{vmatrix} x_c \\ y_c \end{vmatrix} = \begin{vmatrix} -378{,}5765 \\ 549{,}3621 \end{vmatrix} \qquad (3)$$

And its homogeneous coordinates

$$p = \begin{vmatrix} x \\ y \\ w \end{vmatrix} = \begin{vmatrix} -378{,}5765 \\ 549{,}3621 \\ 1 \end{vmatrix} \qquad (4)$$

If now the operator is applied on p:

$$A \times p = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0{,}0016946 & -0{,}29848 \end{vmatrix} \begin{vmatrix} -378{,}5765 \\ 549{,}3621 \\ 1 \end{vmatrix} =$$

$$= \begin{vmatrix} -378{,}58 \\ 549{,}36 \\ 0{,}63247 \end{vmatrix} = \text{`}p \qquad (5)$$

This converts the homogeneous coordinates of the homologous point `P in cartesian coordinates:

$$\text{`}p_c = \begin{vmatrix} x_c \\ y_c \end{vmatrix} = \begin{vmatrix} \frac{x}{w} \\ \frac{y}{w} \end{vmatrix} = \begin{vmatrix} \frac{-378{,}58}{0{,}63247} \\ \frac{549{,}36}{0{,}63247} \end{vmatrix} = \begin{vmatrix} -598{,}57 \\ 868{,}6 \end{vmatrix} \qquad (6)$$

Which are coincidental with those of geometric calculus.

Especial interés ofrece el punto K_3 perteneciente a la recta limite origen k.

$$\text{`}k = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0{,}0016946 & -0{,}2848 \end{vmatrix} \begin{vmatrix} 100 \\ 176{,}1384 \\ 1 \end{vmatrix} = \begin{vmatrix} 100 \\ 176{,}14 \\ 0 \end{vmatrix} \qquad (7)$$

By passing to Cartesian coordinates and finding limits, the coordinates of the homologous point $\text{`}K_3$, are $x = \infty$, $y = \infty$.

From the point of view of computer algorithmic, it is convenient to realize the lost of information associated with the passage to Cartesian coordinates. The homologous point $\text{`}K_{3\infty}$, is an infinite point of the plane, i.e., a direction given by the angle β that can be calculated using the homogeneous coordinates x and y:

$$\beta = \mathrm{tg}^{-1}\frac{176{,}14}{100} = 60{,}415^{o} \qquad (8)$$

As expected, this direction coincides with the direction of the V-ray that goes through the origin point K_3.

The *Perspective matrix projection*, widely used in computer design algorithms, is precisely a particular case of (1). This is equivalent to saying that the conic system of Descriptive Geometry is a particular case of 3D homology when its characteristic is zero, due to $\gamma = 0$.

This provides new procedures for geometric calculations in perspective drawing that eliminate the difficulties related to perpendicularity.

4 Ellipses intersecting in two points

Using the above-mentioned procedures, the four homologies $\theta \Rightarrow `\theta$ are calculated and drawn.

We must consider also using the inversed homologies $`\theta \Rightarrow \theta$ that are found geometrically by permuting the limit lines, and analytically by inversing the transformation matrix.

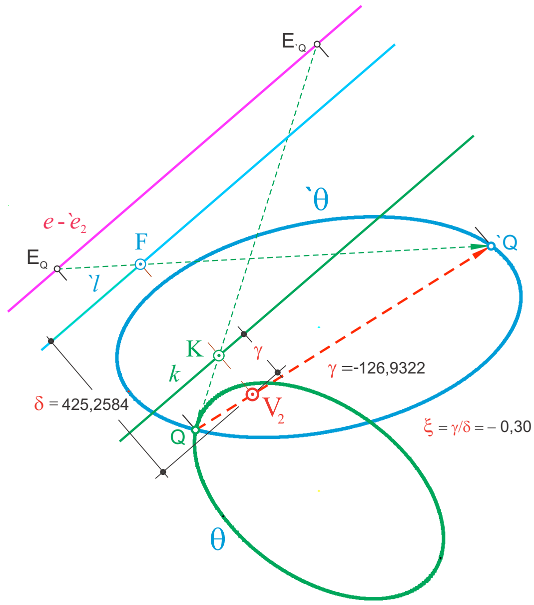


Fig. 17 ξ *= - 0,30*

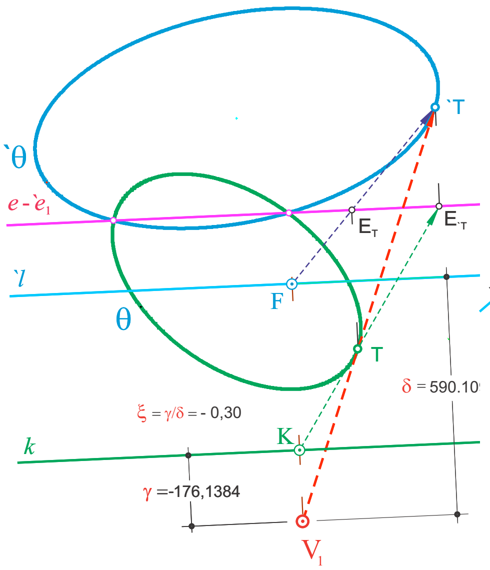


Fig. 15 ξ*= - 0,30*

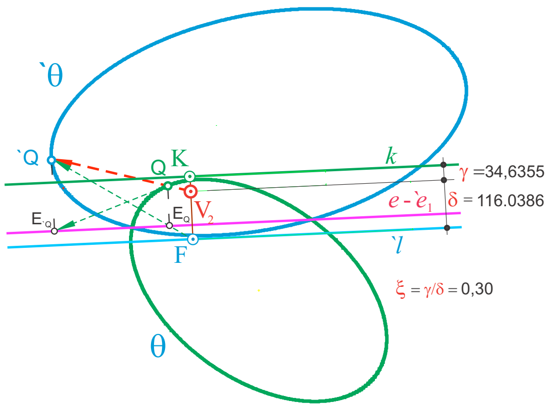


Fig. 16 ξ *= 0,30*

Therefore, we count in this case with eight matrices and eight homologies.

The number of homologies depends on the relative position of both ellipses.

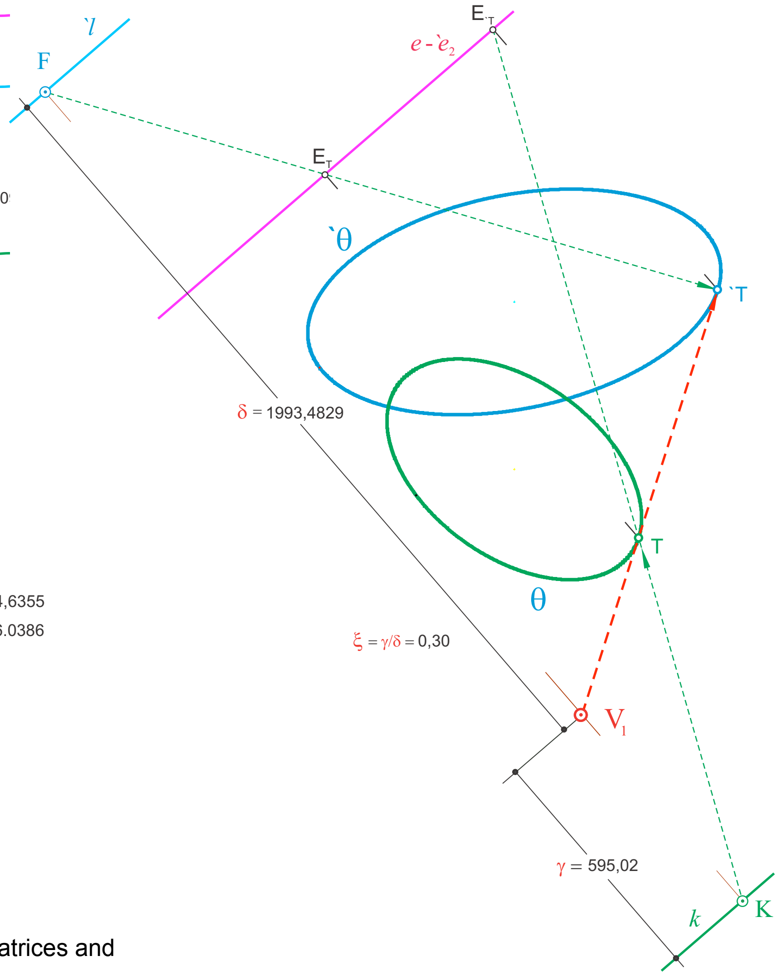


Fig. 18 ξ= 0,30

5 *Internal tangents in a point*

In this case we have six homologies

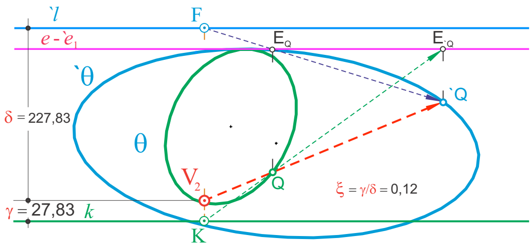


Fig. 19 $\xi = 0{,}12$

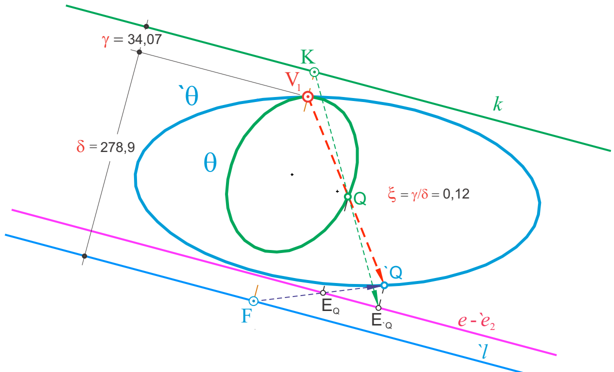


Fig. 20 $\xi = 0{,}12$

Even though their characteristic remains the same and the image figure is identical, their matix are different. They differ in δ. This implies that, after all, using the name 'characteristic' for the transformation determinant has not been the best choice.

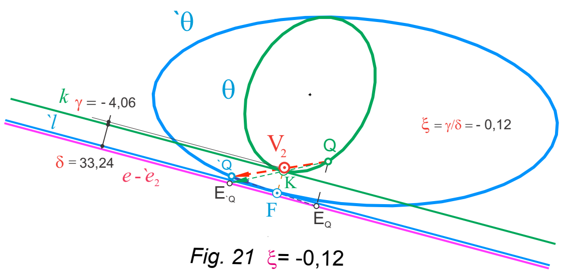


Fig. 21 $\xi = -0{,}12$

θ is not tangent to k, the same way as $`l$ is not tangent to $`\theta$.

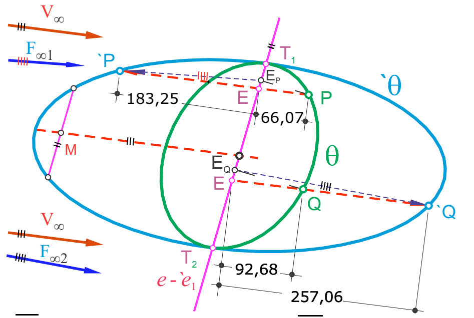
6 *Concentric and internal tangents*

The two first homologies are affinities. Their characteristics are now defined by a simple ratio, as shown in Fig. 22. Their inverse should also be considered, which means that Fig. 22 implies four homologies.

The affinity axis e - $`e_1$ is common to the two affinities, and, as expected, it has the direction of an invariant chord. Geometrically, the affinity is unequivocally defined by the set of three: $V_\infty$, $F_\infty$, $e$ - $`e_1$.

It can be demonstrated (Cfr.: 2 pg 4.1) that the characteristic can be defined by the simple ratio

$$\xi = \frac{\overline{EP}}{\overline{E`P}} \qquad (9)$$



$$\xi_1 = \frac{\overline{EP}}{\overline{E`P}} = \frac{66{,}07}{-183{,}25} = -0{,}36 \qquad \xi_2 = \frac{\overline{EQ}}{\overline{E`Q}} = \frac{92{,}68}{257{,}06} = 0{,}36$$

Fig. 22 Affinities

In Figs. 23 and 24 we show the other two homologies, that carry also their inverse. We count, in this case, with eight homologies.

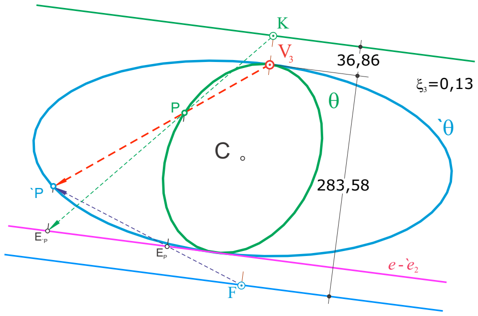


Fig. 23 $\xi_3 = 0{,}13$

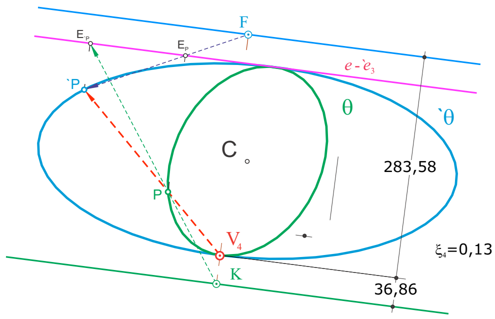


Fig. 24 $\xi_4 = 0{,}13$

A disturbing question that challenges our logical rigor would be: Are they the same homology? The clue of the negative answer is in the sign of parameter δ.

7 *Non concentric internal tangents*

In this case, Fig. 26, we have two homologies and their inverses.

The line linking the tangent points 1-`1 and 2-`2 is the polar of V with respect to θ and $`\theta$. The line containing the conjugated diameter to the polar direction goes through V.

The points 1-`1 belong to the same V-ray and are overlapped. Therefore they belong to the homology axis. Same holds for 2-`2. Therefore, the line determined by both double points is the homology axis e - $`e$.

Aligned poles correspond to polars converging into a point. Thus, any polar of a point of e - $`e$ passes through V; v.g.: 3-'3 and 4-'4 have, respectively, as polars the V-rays T_1T_2 and T_3T_4, whose intersection provides V.

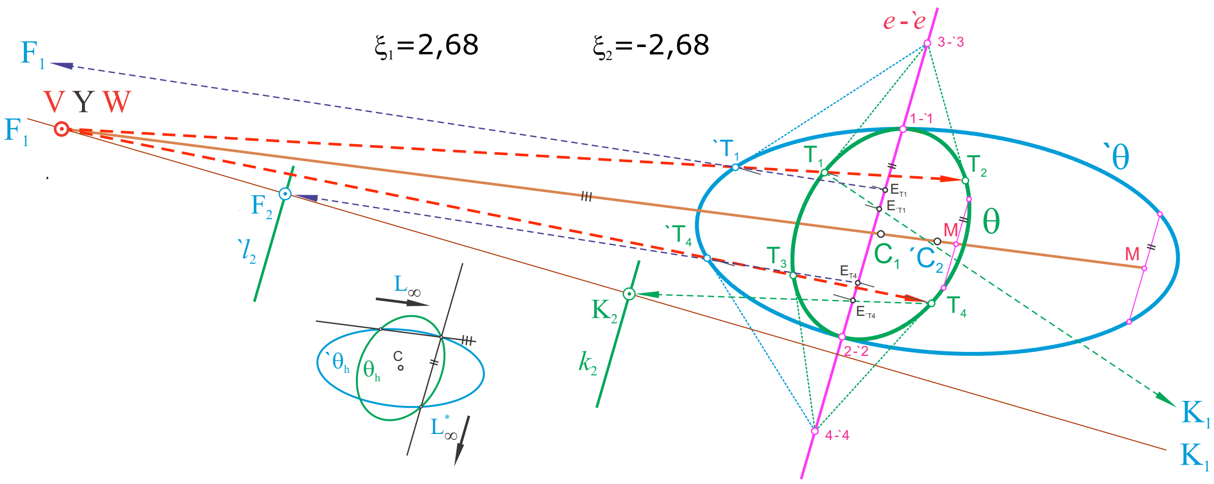


Fig. 25 Two homologies and their inverse ones

Depending upon which pair of homologous points is considered, ‘T_1 and T_1 or ‘T_1 and T_2, one or other homology will be set. See the relevant paper of points L_∞ y L^*_∞.

8 Ellipses intersecting in four points

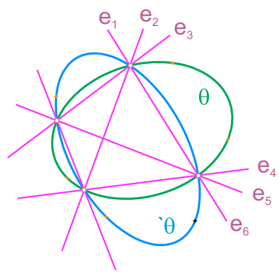


Fig. 26 Possible axes

Every axis determines two homologous e-diameters, d and `d, whose opposite side points determine a couple of homology centers. Not every center-axis combination yield valid homologies.

The homology axes are the sides and the diagonals of a complete four-sided polygon.

The centers of homology are the apex of another four-sided polygon

Every couple of centers that are not co-linear with a third one, determine a co-polar which has two co-poles.

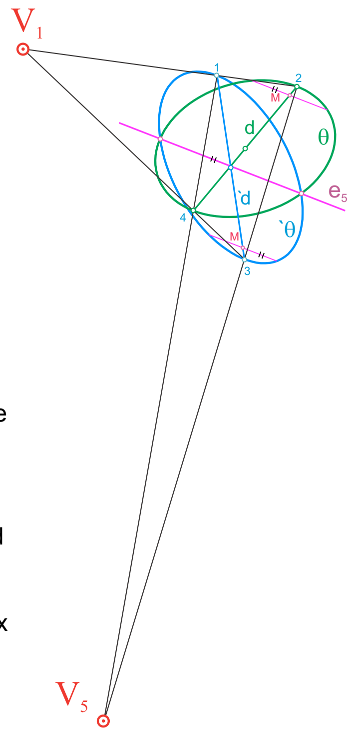


Fig. 27 Center calculation

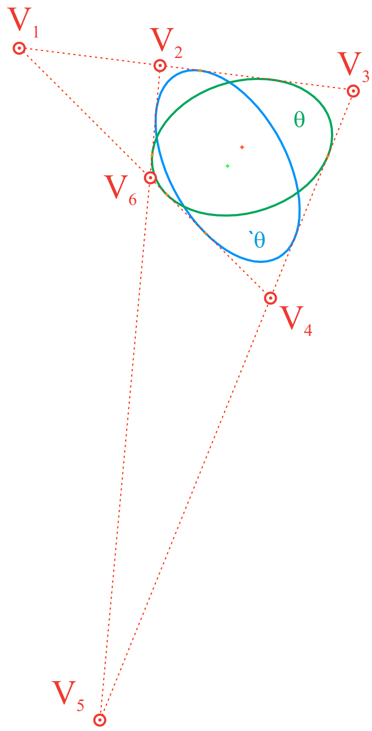


Fig. 28 Possible centers

We count with twelve homologies and another inverse twelve. Therefore, we have twenty four matrices whose characteristics for the proposed ellipses are:

$$\begin{array}{cccccc} 3,27 & 0,27 & 3,27 & 0,27 & 1,14 & 1,14 \\ -3,27 & -0,27 & -3,27 & -0,27 & -1,14 & -1,14 \end{array} \qquad (10)$$

and their inverses.

9 External ellipses

In the examples shown above we have not used any computer algorithm, and the given answers are independent of the problem of tracing tangents to two ellipses.

Furthermore, by finding the homologies linking two ellipses, we solve the problem of tangencies.

An excellent function for the tracing of tangents to two conics is now needed (Autocad).

The tracing of external and internal tangents determines the two centers of homology.

An interesting property of Monge’s circumferences of θ and `θ is that they share the radical axis with the circumference whose diameter is the segment V_1V_2, centered in W.

In Fig. 29 we draw the four homologies θ ⇒`θ. Including their inverses, we have eight matrices.

Point W is the middle point of the segment $V_1 V_2$, which is precisely the point in which the co-polar $V_1 V_2$ intersects the line of centers $`C_2 C_1$
. The other homology elements are found using the procedures shown in Figs. 8 and 9, or in Fig. 12.

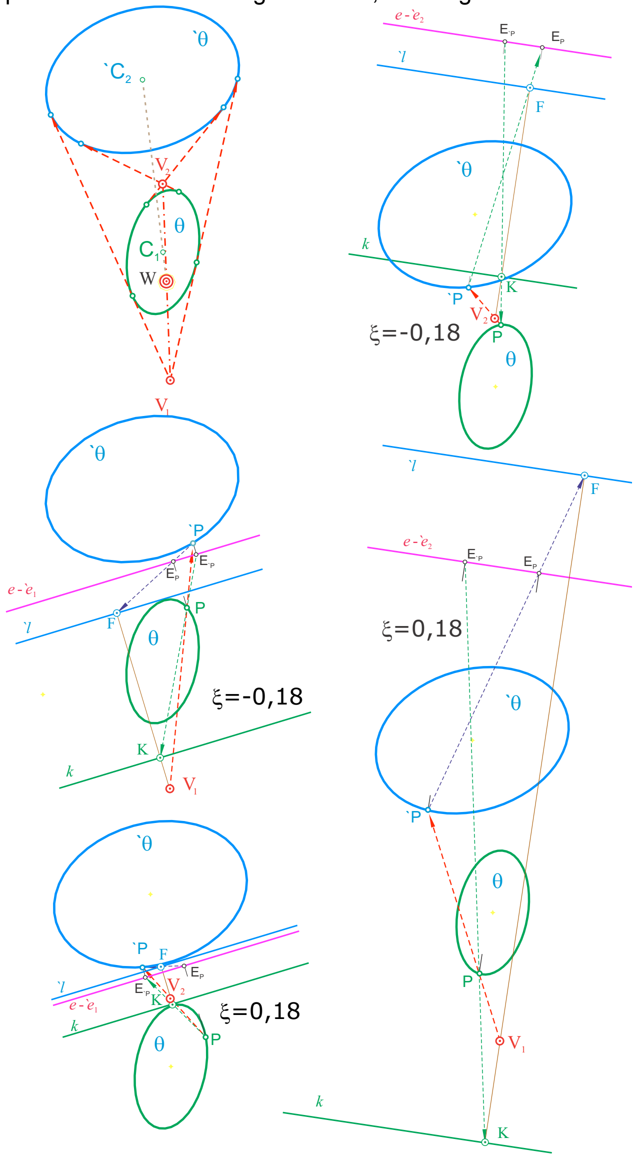


Fig. 29 Centers calculation and homologies

10 Not-solved case

The problem of internal ellipses, without any contact point, is un-resolved. The homologies, drawn a priori, that have to be found are those of Figs. 30 and 31.

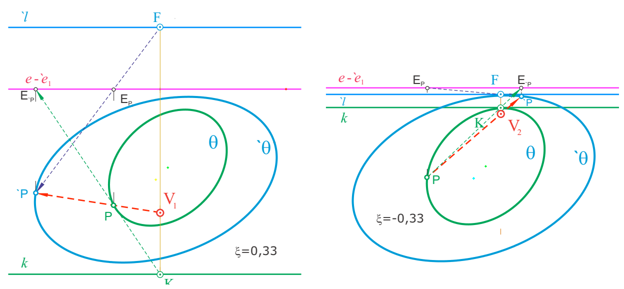


Fig. 30 Unsolved homologies

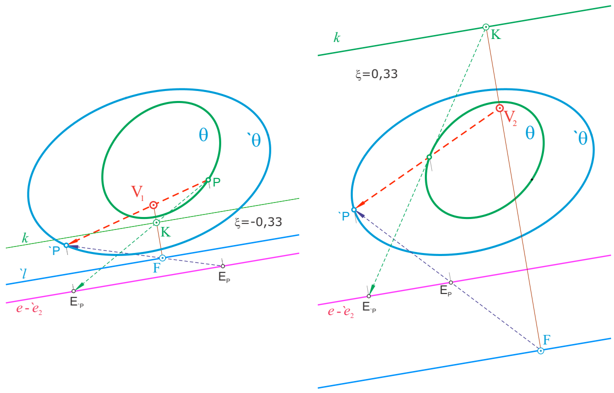


Fig. 31 Unsolved homologies

11 Conical projection of a homology

When we try to find a section generated by a plane in a cone drawn in perspective, we face the difficulty that the elements of spatial homology have lost their parallelism, and, even if the conserve their incidence properties, they are not enough to calculate the section having as origin figure the base of the cone, drawn in perspective.However, the homology, base-section, once drawn, is an alelhomology of two conics that share the same plane: the plane of the paper.The problem is now in determining the elements of this possible homology.

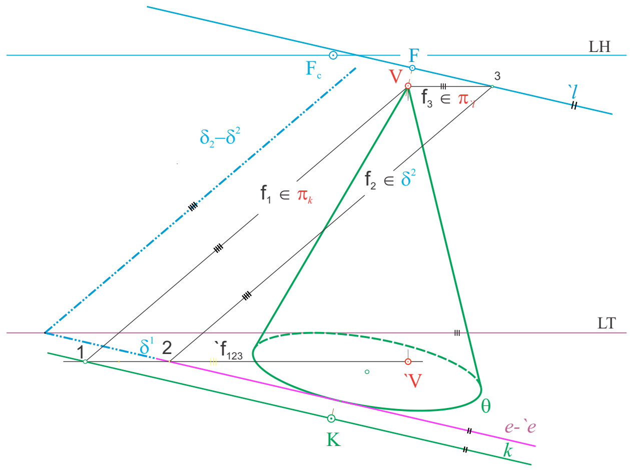


Fig. 32 Homolog elements calculation

The starting data are the drawing of the oblique cone in perspective and the traces of the sectioning plane, δ_1, with the geometral plane and $\delta_2 - \delta_2$ with the paper plane. By the way, double trace n the pass orthogonal to conic, when is adopted for the orthogonal system: horizontal plabe – geometral plane, vertical plane – paper plane and therefore, land line.

The trace δ^1 is the axis of homology *e -`e*.

Through V, center of homology, we trace f_1, the frontal of V-plane π_k, parallel to the sectional plane δ; f_1 goes through V and is parallel to δ^2.

'f_1 goes through 'V and is parallel to LT

$f_1 \cap `f_1 = 1$

Through 1 we trace *k* parallel to δ^1.

$`f_1 \cap e$ - $`e$ = 2

Through 2 we trace f_2 parallel to δ^2.
Through V we trace f_3 parallel to LT.

$f_2 \cap f_3 = 3$

Through 3 we trace $`l$ parallel to $e$ - $`e$
The main V-ray determines F and K

Since we know V and e - $`e$, we could always find F and K using the same procedure used repeatedly, from its point, its homologous and the main V-ray. Its homologous can be calculated from the intersection of the cone generator with the sectional plane.

Now we proceed to the calculation of the section.

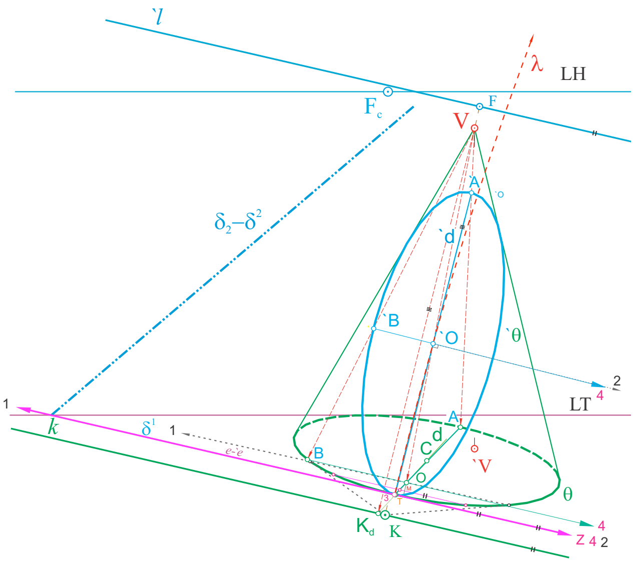


Fig. 33 Calculation of sectional-ellipse

Through the middle point M of a chord parallel to e - $`e$, and through C we trace the e-diameter d that we extend to k, intersecting it in its limit point origin K_d.
Through T, double point of d we trace `d parallel to the V-ray that goes to K_d.

The polar of K with respect to θ dertermines the anti-center O in d and the point 1 in the axis of homology.

The V-ray through O determines, in `d , `O, center of the section ellipse `θ.
Through `O, we trace a perpendicular to `d that determines the point 2 in the axis of homology e - $`e$.

Centered in Z, middle point of the segment 12 and with radius to `O, we draw the circumference λ, that intersects $e$ - $`e$ in the points 3 and 4, double points of axes and anti-axes.
Having traced the anti-axes through 3,4 and O and the axes through 3, 4 and `O, the V-rays through A and B determine the apex `A and `B that, with `O, allow us drawing `θ.

Cfr.: [2], pg. 5.18

In linear-conic perspective the sectional ellipse is not unequivocally determined if its horizontal projection is not drawn.

If we do a horizontal pseudo-projection of the elements of homology of Fig. 33, k and e - $`e$ remain invariant, and only the horizontal projection of $V_1$ and $F_1$. remains to be found. Found V, we can then build by symmetry $`l$ and F.

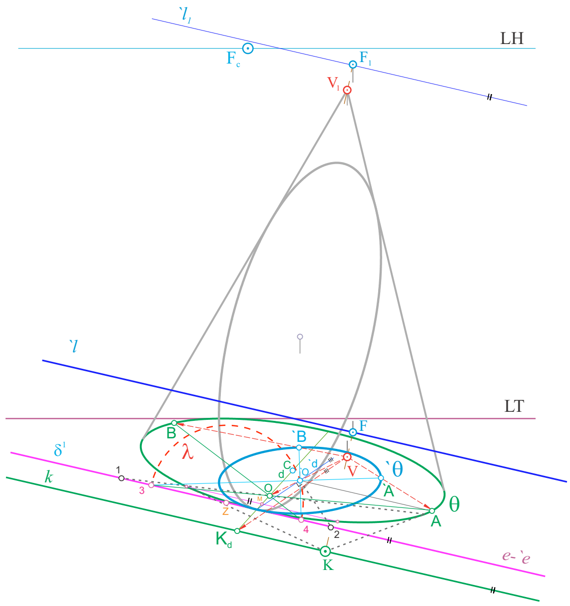


Fig. 34 Homology between base and horizontal projection

Through the middle point M of a chord parallel to e - $`e$ and through C we trace the e-diameter d that we extend until it intersects k in its origin limit point K_d.
Through the d double point we draw `d, parallel to the V-ray that goes to K_d.
The polar K with respect to θ determines the anti-center O in d and the point 1 in the axis of homology.
The V-ray through the anti-center O determines, in `d , `O, which is the center of the ellipse section `θ.
Through `O we trace a perpendicular to `d that determines the point 2 in the homology axis.
With center in Z, middle point of the segment 12 and with radius to `O, we draw the circumference λ, that intersects $e$-$`e$ in the points 3 and 4, double points of axes and anti-axes.
Drawn the anti-axes through 3, 4 and O, and the axis through 3, 4 and `O, the V-rays trough A and B determine the apex `A and `B that, with `O, allows us drawing `θ.

We can proceed in an analogous way when we deal with a section of a cylindrical or prismatic surface.

The center of homology will be now V_∞, which is the direction of generators or edges.

The homology axis e - $`e$ is δ^1, trace with the geometral plane.

To define the affinity it is enough to determine the intersection of the generator that goes through $P' \in \theta$, with δ the sectional plane.

We draw the plane π horizontal projector of the generator.

We find $i = \delta \cap \pi$. The intersection of i with the generator determines point `P.
Through P´ we trace a perpendicular to the homology

axis which determines E_p. The line E_p`P has the direction of F_∞ .

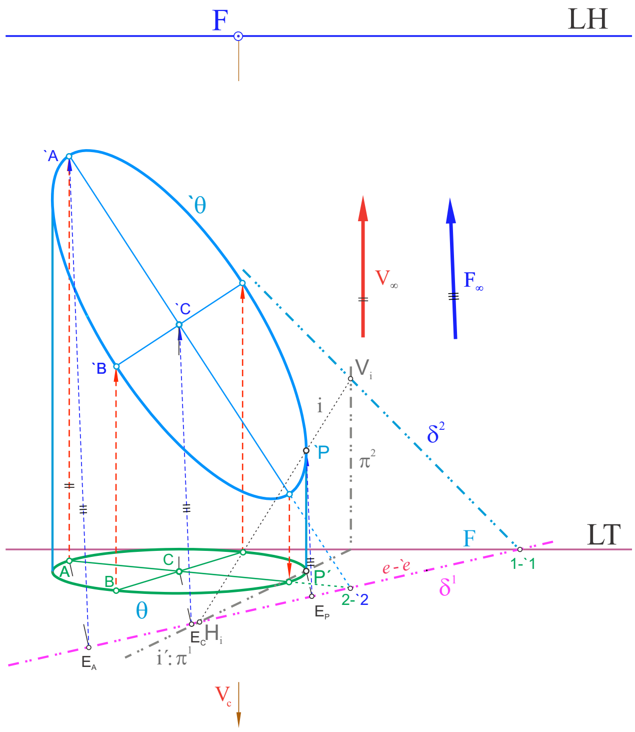


Fig. 35 Base-section affinity

In Fig. 36 the calculation of the affine ellipse is shown.

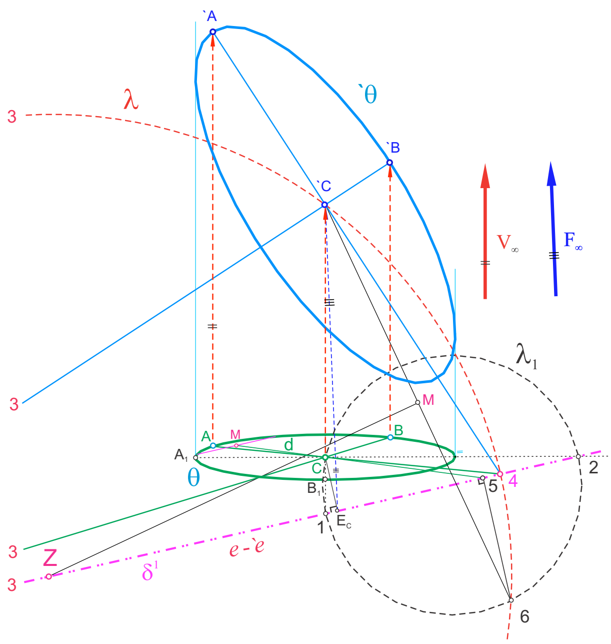


Fig. 36 Calculation of two ellipses related by an affinity

As in an affinity between conics with accessible center the centerness is conserved,

`C = V--ray through C ∩ F-ray through E_C

The axes of theta determine in the points 1 and 2, opposite side point of a diameter of the circumference λ_1 that we draw.

The e-diameter d intersects e-'e in point 5.

A perpendicular to *e* - `*e* through 5 intersects λ_1 in 6.

The perpendicular bisector of the segment `C6 intersects *e* - `*e* in Z.

With center in Z and radius to `C, we draw the circumference λ that intersects *e* - `*e* in the points 3 and 4. These points are double points of axes and anti-axes that go through C and `C.

The V-rays through A and B determine `A and ´B, that allows tracing `θ with `O.

12 Conclusion

We've shown that geometry is not a closed-end science, and that the possibilities for its further development are today greater than ever. People working in the field of design know how much is due to the science of geometry.

It is paradoxical that the more we profit from the fruits of geometry in computer design codes, the less we acknowledge the importance of its contribution.

When will we see software for geometry calculation devoted to the development of research in basic geometry, made with scientific criteria?

We think that much is yet to be done in the field of surface treatment and representation.

To finish, I would like to cite this Dan Pedoe's quote of Henri Lebesgue:

...à une époque où des savants éminents, doués de grands talents géométriques, s`efforçaient de ne jamais dévoiler les idées directes et simples qui les avaient guidès et de faire dèpendre leurs résultats elégants d´une théorie générale abstraite qui, souvent, ne s`appliquait que dans les cas particuliers en question. La géométrie devenait une étude des équations algébriques, differentielles ou aux dérivées partielles: elle perdait ainsi tout le charme qu`elle doit au fait d`etre un art, et presque un art plastique. [3] y [4].

I wish this work has contributed to claim Geometry's beauty and efficiency.

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